Note: Simplex, Tableaus, and LP Duality

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We consider this primal-dual linear program (LP) pair, with the primal in "simplex" form:

$$\max_{x\in\mathbb{R}^n}\{c^\top x : Ax = b, x \geq 0\} \tag{P}$$

and

$$\min_{y\in\mathbb{R}^m}\{b^\top y : A^\top y \geq c\}. \tag{D}$$

Note dimensions of the problem data: $A \in \mathbb{R}^{m\times n}$ with $m \leq n$ (short, fat, underdetermined), $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$.

We partition $\{1, \ldots, n\}$ into the basic index set B and the non-basic index set N, of cardinalities $|B| = m$ and $|N| = n - m$. Without loss of generality we will assume that $B = \{1, \ldots, m\}$ and $N = \{m + 1, \ldots, n\}$.

We have the resulting partitions

$$x = (x_B, x_N), \quad A = [A_B, A_N], \quad c = (c_B, c_N). \tag{1}$$

Supposing that $A_B \in \mathbb{R}^{m\times m}$ is invertible, the primal equality constraints now give

$$A_Bx_B + A_Nx_N = b \iff x_B = A_B^{-1}(b - A_Nx_N). \tag{2}$$

We are solving **this** linear system when we perform row operations until we have an identity block of basic variables from which we can read off the values in the final column of the tableau. If the entries in this column are entirely non-negative, then we have a primal-feasible solution with those values for basic variables, and non-basic variables set to zero. Thus $(x_B, x_N) = (A_B^{-1}b, 0)$.

Now consider the primal objective value. Using our partitions as well as x_B from (2), we write

$$\begin{aligned} c^\top x &= c_B^\top x_B + c_N^\top x_N && (3)\\ &= c_B^\top A_B^{-1}b + \left(c_N^\top - c_B^\top A_B^{-1}A_N\right) x_N. && (4) \end{aligned}$$

The first term here gives us the objective value achieved by our current solution (x_B, x_N) (that is, it determines the bottom right-hand side corner of the tableau). The term in parentheses,

$$\bar{c}_N^\top := c_N^\top - c_B^\top A_B^{-1} A_N, \tag{5}$$

is the vector of "reduced costs" associated with the non-basic variables, which are the entries in the last row of the tableau at non-basic column indices. They are the local derivatives of the objective with respect to the non-basic variables, were we to introduce each of them into the basis.

Now let us turn to the dual problem (D). The dual constraint $A^\top y \geq c$ can be decomposed into

$$A_B^\top y \geq c_B \tag{6}$$
$$A_N^\top y \geq c_N. \tag{7}$$

Since A_B is invertible, an obvious construction for the dual variables can be derived from setting (6) to an equality. Then

$$A_B^\top y = c_B \iff y^\top = c_B^\top A_B^{-1}. \tag{8}$$

Furthermore, this ensures that complementary slackness holds! That is, letting $A_j^\top$ denote the jth row of $A^\top$,

$$x_j \left(A_j^\top y - c_j \right) = 0 \quad \forall j \in \{1, \ldots, n\}, \tag{9}$$

since $x_N = 0$ and $A_B^\top y - c_B = 0$.

In addition to this, dual feasibility requires only that

$$c_N - A_N^\top y \leq 0. \tag{10}$$

Substituting in y from (8), the transpose of the left-hand side of this inequality is

$$c_N^\top - c_B^\top A_B^{-1} A_N = \bar{c}_N^\top, \tag{11}$$

precisely the (row) vector of reduced costs! Therefore, our (implicit) construction of a dual variable y is feasible exactly when the row of reduced costs has only non-positive entries.

As an aside, for these reasons you may sometimes see the vector of reduced costs defined as $\bar{c} := c - A^\top y$.

To recap: we explicitly maintain a primal variable x, associated with an implicit construction of a dual variable y which guarantees complementary slackness. Once the tableau certifies dual feasibility (non-positive reduced costs) and primal feasibility (non-negative basic variables), we have all three conditions of optimality for (P) and (D).